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Propositions and Orthocomplementation in Quantum Logic[1]

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We briefly analyze two partial order relations that are usually introduced in quantum logic by making use of the concepts of "yes–no experiment" and of "preparation" as fundamental. We show that two distinct posets $\mathcal{S}$ and $\mathcal{L}$ can be defined, the latter being identifiable with the lattice of quantum logic. We consider the poset $\mathcal{S}$ and find that it contains a subset $\mathcal{S}_0$ which can easily be orthocomplemented. These results are used, together with suitable assumptions, in order to show that an orthocomplementation in $\mathcal{L}$ can be deduced by the orthocomplementation defined in $\mathcal{S}_0$, and also to give a rule to find the orthocomplement of any element of $\mathcal{L}$.

1. INTRODUCTION

People working on quantum logic have for a long time been accustomed to think that at least two meaningful partial order relations can be given in the set $\mathcal{L}$ of propositions defined for a given physical system, viz., the one used by Mackey (1963),

$$\text{for any } a,b\in\mathcal{L}, \qquad a\leqslant b\Leftrightarrow\alpha(a)\leqslant\alpha(b) \tag{1.1}$$

[where $\alpha(a)$ is the probability of the proposition a when the system is in the state α] and the one used by Jauch (1968) and Piron (1976), which we express here in the equivalent form (Beltrametti and Cassinelli, 1979)

$$\text{for any } a,b\in\mathcal{L}, \qquad a\preccurlyeq b\Leftrightarrow S_1(a)\subseteq S_1(b) \tag{1.2}$$

where $S_1(a)$ [respectively, $S_1(b)$] is the set of the states which make a (respectively, b) certainly true.

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Hence, the problem arises immediately whether these partial order relations coincide in $\mathfrak{L}$. Making suitable assumptions it can be proved that they do (Pool, 1968a); alternatively, one can a priori assume that $\preccurlyeq$ and $\leqslant$ coincide (Beltrametti and Cassinelli, 1976). Nevertheless, this coincidence can be questioned, for instance by making use of Mielnik's analysis and examples (Mielnik, 1976). Moreover, Mielnik's analysis throws further doubts on the orthocomplementation in $\mathfrak{L}$.

To overcome these difficulties one can observe that Mielnik's objections refer to an interpretation of the propositions of $\mathfrak{L}$ as equivalence classes of all the possible yes–no experiments, and that they actually make it evident that the axioms of quantum logic, when interpreted in terms of properties of the yes–no experiments contained in the classes that are the elements of $\mathfrak{L}$, imply that only a limited number of yes–no experiments in any equivalence class can be taken as representative of the corresponding proposition; thus, one can say, the experiments proposed by Mielnik do not belong to this privileged representative set, and their "anomalous" properties must not be considered.

However, this point of view leaves us disappointed, essentially because it does not give an explicit rule to single out the questions permitted in the set of all the physically conceivable yes–no experiments (one can say that they must correspond to ideal, pure, first kind measurement; however, it is not easy to verify if a given yes–no experiment satisfies this condition); moreover, we would prefer an approach that does not eliminate a large set of experimental devices right from the beginning. In the present work, we set up a possible basis for such an approach; we show that the relations $\leqslant$ and $\preccurlyeq$ give rise to different equivalence classes in the set of all the yes–no experiments and obtain two partially ordered sets that we call $\mathfrak{S}$ and $\mathfrak{L}$, this last being identifiable with the lattice of quantum logic. Then, some immediate properties of $\mathfrak{S}$ are discussed (in particular, the existence in $\mathfrak{S}$ of an orthocomplemented subset $\mathfrak{S}_0$) and some axioms are proposed which allow us to transfer properties of $\mathfrak{S}$ (in particular, orthocomplementation) into properties of $\mathfrak{L}$. In our opinion this permits a clearer interpretation of some of the axioms which are usually stated in quantum logic and gives an explicit rule to pin-point the experiments which can be considered representative of a given proposition of $\mathfrak{L}$. Furthermore, the whole discussion turns our attention to a new structure, the one we call $\mathfrak{S}_0$ in Section 3, and one might wonder if it has an independent physical or logical meaning (we do not want to dwell upon this point here).

2. EQUIVALENCE AND ORDER RELATIONS

Following Beltrametti and Cassinelli (1979), we shall consider the concepts of *physical system*, *yes–no experiment* [which we also call "*ques-*

tion" according to Piron (1976)], and *preparation* of a given physical system as the primitive objects of our theory. [Instead of questions we could take observables as primitive objects. We have chosen questions in order to avoid unnecessary complications in this work; however these choices are substantially equivalent (Maczynski, 1972; Beltrametti and Cassinelli, 1979).]

We will denote the set of all the questions by E and the set of all the preparations by Π. We assume that it is physically meaningful to assign a probability to the yes outcome for any yes–no experiment $e \in E$ which is performed on the system prepared according to a given $\pi \in \Pi$ (this is actually a restriction on E), and denote it by $\pi(e)$.

For any π_1, $\pi_2 \in \Pi$ we will say that π_1 has the relation $\equiv$ to π_2 whenever $\pi_1(e) = \pi_2(e)$ for any $e \in E$. The binary relation $\equiv$ is obviously an equivalence relation in Π. We will call the equivalence classes $[\pi]_{\equiv}$, $\pi \in \Pi$, *states* of the system, and denote the quotient set $\Pi/_{\equiv}$ by S. For any $\alpha \in S$ we set $\alpha(e) = \pi(e)$, with $\pi \in \alpha$, and say that $\alpha(e)$ is the probability of the yes outcome for the yes–no experiment e when the system is in the state α. Then, we introduce a binary relation $\leqslant$ in E by setting

$$\text{for any } e_1, e_2 \in E, \qquad e_1 \leqslant e_2 \Leftrightarrow \text{for any } \alpha \in S,\ \alpha(e_1) \leqslant \alpha(e_2) \tag{2.1}$$

This relation is reflexive and transitive; moreover, it is easy to imagine some yes–no experiments e_1, e_2, such that $\alpha(e_1) = \alpha(e_2)$ with $e_1 \neq e_2$, so that $\leqslant$ is not antisymmetric. In order to obtain a partially ordered set, we introduce a further binary relation in E by setting

$$\text{for any } e_1, e_2 \in E, \qquad e_1 \sim e_2 \Leftrightarrow \text{for any } \alpha \in S,\ \alpha(e_1) = \alpha(e_2) \tag{2.2}$$

(hence $e_1 \sim e_2 \Leftrightarrow e_1 \leqslant e_2$ and $e_2 \leqslant e_1$). This is an equivalence relation in E; thus, we can consider the quotient set $\mathfrak{E} = E/\sim$. Let $x \in \mathfrak{E}$, $e \in x$; we put, for any $\alpha \in S$, $\alpha(x) = \alpha(e)$. Hence, a binary relation, which we again denote by $\leqslant$, can be introduced in $\mathfrak{E}$ by setting

$$\text{for any } x_1, x_2 \in \mathfrak{E}, \qquad x_1 \leqslant x_2 \Leftrightarrow \text{for any } \alpha \in S,\ \alpha(x_1) \leqslant \alpha(x_2) \tag{2.3}$$

Now, $\leqslant$ is obviously a partial order relation in $\mathfrak{E}$.

Our procedure can be repeated with a different kind of binary relation in E. More precisely, for any $e \in E$, we put $S_1(e) = \{\alpha \in S\colon \alpha(e) = 1\}$ (this set will be called the *certainly yes domain* of e) and introduce the binary relation $\preccurlyeq$ in E by setting

$$\text{for any } e_1, e_2 \in E, \qquad e_1 \preccurlyeq e_2 \Leftrightarrow S_1(e_1) \subseteq S_1(e_2) \tag{2.4}$$

[Here $\subseteq$ denotes set inclusion. We recall that the set of states of S, hence of $S_1(e)$, can be divided into pure states and mixtures; both kinds of states actually exist in classical and in quantum mechanics. However, in quantum logic $\preccurlyeq$ could also be defined by making reference to pure states only; this would not alter the essential features of our argument.] Again, this relation is reflexive and transitive but not antisymmetric; as before, we can define another binary relation in E by setting

$$\text{for any } e_1, e_2 \in E, \qquad e_1 \approx e_2 \Leftrightarrow S_1(e_1) = S_1(e_2) \tag{2.5}$$

(hence $e_1 \approx e_2 \Leftrightarrow e_1 \preccurlyeq e_2$ and $e_2 \preccurlyeq e_1$). This is an equivalence relation; thus, we consider the quotient set $\mathfrak{L} = E/\approx$ and for any $a \in \mathfrak{L}$ we put $S_1(a) = S_1(e)$, with $e \in a$. Hence a binary relation, which we again denote by $\preccurlyeq$, can be introduced in $\mathfrak{L}$ by setting

$$\text{for any } a_1, a_2, \qquad a_1 \preccurlyeq a_2 \Leftrightarrow S_1(a_1) \subseteq S_1(a_2) \tag{2.6}$$

This is obviously a partial order relation in $\mathfrak{L}$ [we remember (Beltrametti and Cassinelli, 1979; Pool, 1968a) that the order relation (2.6) rephrases the order relation introduced by Jauch (1968), Piron (1976), and others by making use of the concept of preparation as a primitive concept] and say that these propositions also form a partially ordered set.

Now, the problem arises whether the elements of $\mathfrak{S}$ coincide with the elements of $\mathfrak{L}$. There are examples of yes–no experiments proposed by Mielnik (1976) which show that it is possible to imagine some yes–no experiments $e_1, e_2 \in E$ such that $e_1 \approx e_2$ but $e_1 \not\sim e_2$. Thus, we conclude that the sets $\mathfrak{S}$ and $\mathfrak{L}$ have different elements, so that it is meaningless to ask whether the order relations $\leqslant$ and $\preccurlyeq$ coincide. However, our analysis can be deepened further.

Indeed, we observe that

$$\text{for any } e_1, e_2 \in E, \qquad e_1 \leqslant e_2 \Rightarrow e_1 \preccurlyeq e_2 \tag{2.7}$$

hence $e_1 \sim e_2 \Rightarrow e_1 \approx e_2$. Thus, for any $e \in E$, $[e]_\sim \subseteq [e]_\approx$. [Equivalently, denoting the partial order relation usually introduced in the set $\mathfrak{R}(E)$ of the equivalence relations in E again by $\subseteq$, we can say that $\sim \subseteq \approx$.] Then, the quotient relation $\approxeq = \overset{\approx}{\sim}$ can be introduced in $\mathfrak{S} = E/\sim$ by setting (Bourbaki, 1966)

$$\text{for any } x_1 = [e_1]_\sim \in \mathfrak{S} \quad \text{and} \quad x_2 = [e_2]_\sim \in \mathfrak{S}$$

$$x_1 \approxeq x_2 \Leftrightarrow [e_1]_\approx = [e_2]_\approx \tag{2.8}$$

This is again an equivalence relation, so that the quotient set $\mathfrak{L}'=\mathfrak{S}/\approx$ can be considered. Let $e\in E$, $x=[e]_{\sim}$, $a'=[x]\approx$; then, obviously, the set of all the yes–no experiments belonging to the elements $y\in\mathfrak{S}$ such that $y\in a'$ is $[e]_{\approx}$, so that the mapping

$$\phi\colon a'=[x]\approx\to a=[e]_{\approx}$$

is bijective. Hence, the sets $\mathfrak{L}'$ and $\mathfrak{L}$ can be identified; we will make this identification from now on, so that we can say that every element of $\mathfrak{L}$ is an equivalence class in the set $\mathfrak{S}$.

Furthermore, we introduce a new binary relation in $\mathfrak{L}$, which we again denote by $\leqslant$, by setting

$$\text{for any } a_1, a_2\in\mathfrak{L}$$

$$a_1\leqslant a_2 \Leftrightarrow \text{two questions } e_1\in a_1, e_2\in a_2 \text{ exist such that } e_1\leqslant e_2 \tag{2.9}$$

(hence $a_1\leqslant a_2 \Rightarrow a_1 \preccurlyeq a_2$).

This relation is reflexive, transitive, and antisymmetric [because of (2.7) and the antisymmetry of $\preccurlyeq$ in $\mathfrak{L}$]; thus, $\mathfrak{L}$ is partially ordered by $\leqslant$ and the question arises whether $\leqslant$ and $\preccurlyeq$ coincide in $\mathfrak{L}$; since it seems that no purely logical reason exists to decide in either way, it is possible to introduce this coincidence as a new assumption of the theory. We refer to Beltrametti and Cassinelli (1976) for such an assumption, only noticing that the Axiom 5 introduced in Section 2.1 of this reference must, in our opinion, be reinterpreted in the sense discussed above.

3. FIRST INVESTIGATION OF $\mathfrak{S}$

We assume that a "certainly true" and a "certainly false" experiment, which we denote by e_I and by e_0, respectively, exist in E (these are idealized experiments whose meaning is obvious) and denote the corresponding elements of $\mathfrak{S}$ by I and 0, respectively; thus, I and 0 are the greatest and the least elements of $\mathfrak{S}$.

For any $e\in E$, we consider the experiment $e^{\perp}$ obtained from e by interchanging the yes and no outcomes.

It is apparent that

$$\text{for any } \alpha\in S, \qquad \alpha(e^{\perp})=1-\alpha(e) \tag{3.1}$$

and that

$$\text{for any } e_1, e_2\in E, \qquad e_1\sim e_2 \Leftrightarrow e_1{}^{\perp}\sim e_2{}^{\perp} \tag{3.2}$$

Thus, we can assume that, for any $x \in \mathfrak{S}$, an element $x^{\perp} \in \mathfrak{S}$ exists such that $\alpha(x^{\perp}) = 1 - \alpha(x)$ for any $\alpha \in S$. Moreover, the mapping η: $x \in \mathfrak{S} \rightarrow x^{\perp} \in \mathfrak{S}$ is such that

$$\text{for any } x \in \mathfrak{S}, \qquad x^{\perp\perp} = x \tag{3.3}$$

$$\text{for any } x,y \in \mathfrak{S}, \qquad x \leqslant y \Rightarrow y^{\perp} \leqslant x^{\perp} \tag{3.4}$$

However, $x^{\perp}$ is not, in general, a complement of x, as simple examples can easily show. For instance, let e be a semitransparent mirror and let $x = [e]_{\sim}$; then, $x^{\perp} = x$ and the join $x \vee x^{\perp} = x$ exists in $\mathfrak{S}$ and does not coincide with I.

Because of the outstanding importance of the orthocomplementation in quantum logic, we wonder if η can become an orthocomplementation by eliminating a proper class of annoying experiments from E. Indeed, the way of doing this exists and is very simple. Let us consider the subset $E_0 \subseteq E$ such that a yes–no experiment $e \in E$ belongs to E_0 if and only if $e \nleqslant e^{\perp}$ and $e \ngeqslant e^{\perp}$ (briefly, $e \| e^{\perp}$), the only exceptions being e_0 and e_I, which we always admit in E_0. Then, the binary relations $\leqslant$ and $\sim$ are defined in E_0 by restriction of the relations $\leqslant$ and $\sim$ defined in E (this can also be done with $\preccurlyeq$ and $\approx$, which, however, do not interest us here); thus, we can consider the quotient set $\mathfrak{S}_0 = E_0/\sim$. We observe that $\mathfrak{S}_0$ is actually a subset of $\mathfrak{S}$; indeed, in order to get E_0 from E, every equivalence class $[e]_{\sim}$ must either be entirely eliminated or entirely accepted in E_0 because of (3.2). Moreover, being trivially $e \in E_0 \Rightarrow e^{\perp} \in E_0$, for any $x \in \mathfrak{S}_0, x^{\perp} \in \mathfrak{S}_0$; then, $x \nleqslant x^{\perp}$ and $x^{\perp} \nleqslant x$ (briefly $x \| x^{\perp}$), with the only exceptions of 0 and I (which obviously belong to $\mathfrak{S}_0$).

We notice that, for any $e \in E$, the condition $e \| e^{\perp}$ is clearly equivalent to the following:

Condition C. a pair (α, β) of states exists in S such that

$$\alpha(e) < \tfrac{1}{2} \quad \text{and} \quad \beta(e) > \tfrac{1}{2}$$

Condition C is fulfilled by most meaningful yes–no experiments. For instance every question e such that at least one state α with $\alpha(e) = 1$ and one state β with $\beta(e) = 0$ exists (this seems a reasonable requirement if we want e to yield significant physical information) satisfies it.

We can now state the following proposition.

Proposition 1. The restriction to $\mathfrak{S}_0$ of the mapping η: $x \in \mathfrak{S} \rightarrow x^{\perp} \in \mathfrak{S}$ is an orthocomplementation in $\mathfrak{S}_0$.

Proof. The join $x\vee x^{\perp}$ exists in $\mathfrak{S}_0$ and coincides with I. Indeed, let $e,f\in E_0$ and let $f\geqslant e, f\geqslant e^{\perp}$; it follows $\alpha(f)\geqslant\frac{1}{2}$ for any $\alpha\in S$. Thus, f violates condition C, so that it must be equivalent to e_I, and $[f]_{\smile}=I$; hence our statement about $x\vee x^{\perp}$ follows immediately.

Analogously, we can show that, for any $x\in\mathfrak{S}_0$, the meet $x\wedge x^{\perp}$ exists in $\mathfrak{S}_0$ and coincides with 0. Thus, recalling (3.3) and (3.4), we see that η is an orthocomplementation in $\mathfrak{S}_0$. ∎

4. THE ORTHOCOMPLEMENTATION IN $\mathfrak{L}$

If one identifies $\mathfrak{L}$ with the set of the propositions of quantum logic, this identification may cause some trouble, as we have already pointed out in the Introduction. Indeed, in quantum logic, $\mathfrak{L}$ is a complete, orthomodular, atomic lattice which obeys the covering law; then, some of these properties can be questioned (Mielnik, 1976) if interpreted in terms of properties of the yes–no experiments contained in the elements of $\mathfrak{L}$.

On the other hand, together with some new suitable assumptions, our present knowledge of $\mathfrak{S}$ can be used to deduce further information about $\mathfrak{L}$; we shall be particularly concerned here with the orthocomplementation in $\mathfrak{L}$.

Firstly, we observe that the equivalence classes $\phi=[e_0]_{\approx}$ and $I=[e_I]_{\approx}(=[e_I]_{\smile})$ are the least and the greatest elements of $\mathfrak{L}$, respectively. Then, we can assume that for any $a\in\mathfrak{L}$ an orthocomplement $a^{\perp}\in\mathfrak{L}$ exists (Jauch, 1968) or we can give equivalent axioms (Piron, 1976); nevertheless, with this assumption one does not give a clear rule for which yes–no experiments actually belong to $a^{\perp}$. [It has been said by Jauch (1968) and in some early papers by Piron (1964) that $a^{\perp}$ contains all the yes–no experiments obtained by the ones in a by reversing the roles of yes and no. We have seen that this is a good recipe for obtaining an orthocomplement in $\mathfrak{S}_0$, but it is untenable in $\mathfrak{L}$ because of Mielnik (1976).] Unfortunately, the procedure that we have followed in order to obtain a "natural" notion of orthocomplementation in $\mathfrak{S}_0$ cannot be repeated here because of a basic difference between $\mathfrak{S}$ and $\mathfrak{L}$, i.e., because an analogue of (3.2) does not hold in $\mathfrak{L}$. In fact, let $e_1,e_2\in E$; then, $e_1\approx e_2$ does not imply $e_1{}^{\perp}\approx e_2{}^{\perp}$. Actually, some yes–no experiments can be imagined (Mielnik, 1976) such that $e_1\approx e_2$ but $e_1{}^{\perp}\not\approx e_2{}^{\perp}$. (With the purpose of recovering an orthocomplementation in $\mathfrak{L}$ we could say that we eliminate the arrangements that cause trouble from E. However, we think that this solution is unsatisfactory, if some rule is not given to select the accepted yes–no experiments.) To overcome these difficulties we put, for any $e\in E$, $S_0(e)=S_1(e^{\perp})$ [this will be called the *certainly not domain* of e; obviously, $S_0(e)=\{\alpha\in S$:

$\alpha(e)=0\}]$ and introduce the following assumptions:

Axiom 1. For any $a \in \mathcal{L}$, at least one $e_a \in E$, $e_a \in a$ exists with maximum "certainly not" domain, i.e., such that $S_0(e_a)$ is maximum, with respect to set inclusion, in the set $\{S_0(e): e \in a\}$.

Axiom 2. For any $a, b \in \mathcal{L}$, let $e_a \in a, e_b \in b$ be yes–no experiments with maximum "certainly not" domain; then

$$e_a \not\sim e_b \Rightarrow S_0(e_a) \neq S_0(e_b)$$

Axiom 3. Let $a, b \in \mathcal{L}, a \preccurlyeq b$. Then some yes–no experiments $e_a \in a, e_b \in b$ with maximum "certainly not" domains exist such that $e_a \leqslant e_b$.

Before going on we would like to comment on these axioms briefly.

Axiom 1 and Axiom 2 are consistent with the usual formulation of quantum mechanics if we think of $a \in \mathcal{L}$ as a subspace in the lattice $\mathcal{L}(\mathcal{H})$ of the closed subspaces of the Hilbert space $\mathcal{H}$ of the system; indeed, if we admit that for any a at least one experimental arrangement e_a exists which gives the yes outcome if and only if a is certainly true and the no outcome if and only if a is certainly false, then Axioms 1 and 2 immediately follow. [We remember that an assumption similar to Axiom 1 has been suggested by Mielnik (1969, 1974) by selecting in E only questions with maximal "certainly not" domain.]

Axiom 3 essentially states [if one recalls (2.7)] that $\preccurlyeq$ and $\leqslant$ coincide in $\mathcal{L}$ [$\leqslant$ is defined in $\mathcal{L}$ by (2.9)]. This coincidence is assumed (although without a clear distinction between $\mathcal{E}$ and $\mathcal{L}$) in some axiomatic approaches to quantum logic (Beltrametti and Cassinelli, 1976), while it is derived as a consequence of suitable axioms in other approaches (Pool, 1968a). (In this last reference the propositions of $\mathcal{L}$ are called events and the set of the events and the set of the states are chosen as primitive objects; whenever the events are interpreted as equivalence classes of yes–no experiments the properties of the events are shared by a limited number of the questions which form them according to our previous discussion. We can call these privileged questions "observation procedures" and identify them with the yes–no experiments with maximum "certainly not" domain whose existence is postulated by Axiom 1.)

Coming back to our problem, we first notice that it follows from Axiom 1 that at least one element $x_a = [e_a]_\sim$ such that the yes–no experiments belonging to it have a maximum "certainly not" domain corresponds to each $a \in \mathcal{L}$. Moreover, x_a actually belongs to $\mathcal{E}_0$; indeed, for any $a \neq \phi$, $S_1(e_a)$ is nonvoid and, for any $a \neq I$, $S_0(e_a)$ is also nonvoid, because of Axiom 2, so that any question $e_a \in x_a$ fulfills condition C of Section 2 whenever $\phi \neq a \neq I$. Finally, it follows from Axiom 2 that, for any $a,b \in \mathfrak{L}$, the yes–no experiments $e_a \in a, e_b \in b$ with maximum "certainly not" domain are such that $a = b \Rightarrow e_a \sim e_b$; hence, the element $x_a = [e_a]_\sim$ is unique for any $a \in \mathfrak{L}$.

Thus, by making use of Axiom 3, we get that the mapping which makes $a \in \mathfrak{L}$ correspond to $x_a \in \mathfrak{S}$ is an isotonic injection of $\mathfrak{L}$ into $\mathfrak{S}_0$. In the sequel, we will denote the range of this injection by $\mathfrak{S}_\mathfrak{L}$; then, recalling (2.7), we see that the mapping g: $a \in \mathfrak{L} \to x_a \in \mathfrak{S}_\mathfrak{L}$ is an order isomorphism between $\mathfrak{L}$ and $\mathfrak{S}_\mathfrak{L} \subseteq \mathfrak{S}_0$.

For any $a \in \mathfrak{L}$, let us set $a^\perp = [e_a{}^\perp]_\approx$: this proposition is easily seen to be uniquely determined in $\mathfrak{L}$. Now we can state the following proposition.

Proposition 2. Let $a \in \mathfrak{L}$; then, the mapping θ: $a \in \mathfrak{L} \to a^\perp \in \mathfrak{L}$ is an orthocomplementation in $\mathfrak{L}$.

Proof. Let $x^\perp$ be the orthocomplement in $\mathfrak{S}_0$ of any $x \in \mathfrak{S}_0$, and let $a \in \mathfrak{L}, x_a = [e_a]_\sim \in \mathfrak{S}_\mathfrak{L}$. Then, $x_a{}^\perp = [e_a{}^\perp]_\sim \in \mathfrak{S}_\mathfrak{L}$ so that $\mathfrak{S}_\mathfrak{L}$ is orthocomplemented. Indeed, let us consider $[e_a{}^\perp]_\approx = a^\perp \in \mathfrak{L}$; because of Axiom 1 an element $e_{a^\perp} \in E$ exists whose "certainly not" domain is maximum. Let us suppose that $e_{a^\perp} \not\sim e_a{}^\perp$; then, since $e_a{}^\perp \in a^\perp$, $e_{a^\perp} \in a^\perp$, $S_0(e_{a^\perp}) \supset S_0(e_a{}^\perp)$ (here, $\supset$ means $\supseteq$ with the exclusion of equality); hence, $e_a{}^{\perp\perp} = e_a \preccurlyeq e_{a^\perp}{}^\perp$ and $e_a{}^{\perp\perp} = e_a \not\approx e_{a^\perp}{}^\perp$. Thus, by making use of Axioms 2 and 3 we obtain $S_0(e_a) \supset S_0(e_{a^\perp}{}^\perp)$ while, by construction, $S_0(e_a) = S_0(e_{a^\perp}{}^\perp)$. Therefore our assumption $e_{a^\perp} \not\sim e_a{}^\perp$ leads to a contradiction, hence $e_{a^\perp} \sim e_a{}^\perp$, i.e., $x_a{}^\perp = [e_a{}^\perp]_\sim \in \mathfrak{S}_\mathfrak{L}$, as stated.

Thus, bearing in mind that the mapping g: $a \in \mathfrak{L} \to x_a \in \mathfrak{S}_\mathfrak{L}$ is an order isomorphism, the statement of our proposition follows easily. ■

To sum up the results achieved in this section, we can say that we have proved that an orthocomplementation in $\mathfrak{L}$ can be given starting from the orthocomplementation defined in $\mathfrak{S}_0$, by making use of suitable assumptions. Moreover, we have implicitly obtained a rule to find the orthocomplement of any $a \in \mathfrak{L}$, which in explicit terms sounds as follows: "Consider an $a \in \mathfrak{L}$; choose a yes–no experiment e_a in a which has a maximum "certainly not" domain; consider the yes–no experiment $e_a{}^\perp$ obtained by e_a by reversing the roles of the yes and no outcomes; consider the proposition $a^\perp \in \mathfrak{L}$ such that $e_a{}^\perp \in a^\perp$; then $a^\perp$ is the orthocomplement of a." We also observe that this orthocomplement is a compatible complement in the sense of Piron (1976). Of course, the above construction of the orthocomplement rests on the assumption that physics allows us to pick out a yes–no experiment with maximum "certainly not" domain between the questions pertaining to any proposition in $\mathfrak{L}$.

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